The interplay between circle packings and subdivision rules

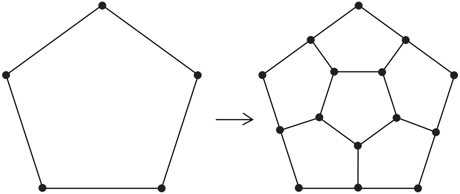
Bill Floyd (joint work with Jim Cannon and Walter Parry)

Virginia Tech

July 9, 2020

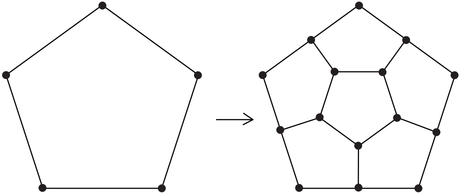
A preexample, the pentagonal subdivision rule

Can you recursively subdivide pentagons by this combinatorial rule so that the shapes stay almost round (inscribed and circumscribed disks with uniformly bounded ratios of the radii) at all levels?

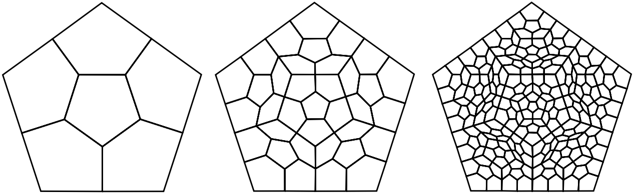


A preexample, the pentagonal subdivision rule

Can you recursively subdivide pentagons by this combinatorial rule so that the shapes stay almost round (inscribed and circumscribed disks with uniformly bounded ratios of the radii) at all levels?

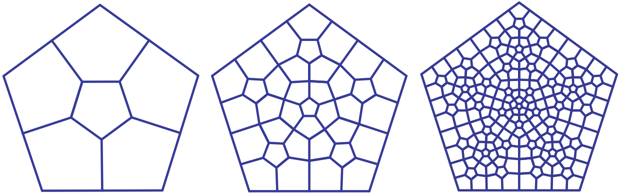


Subdividing them affinely doesn't work.



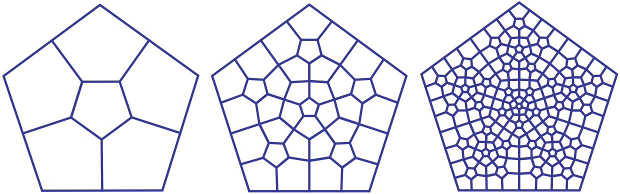
Generating the figures with CirclePack

In 1994, Ken Stephenson suggested using CirclePack to draw the tilings.



Generating the figures with CirclePack

In 1994, Ken Stephenson suggested using CirclePack to draw the tilings.



It follows from the Ring Lemma of Rodin-Sullivan that the pentagons will be almost round at each level. But the three figures are produced independently. What is impressive is how much they look like subdivisions.

Definition of a finite subdivision rule $\mathcal{R}$

- C-F-P, Finite subdivision rules, *Conform. Geom. Dyn.* **5** (2001) 153–196
- finite CW complex $S_{\mathcal{R}}$ (called the *model subdivision complex*)
- $S_{\mathcal{R}}$ is the union of its closed 2-cells. Each 2-cell is modeled on a polygon (called a *tile type*). The 1-cells in $S_{\mathcal{R}}$ are called *edge types*.
- subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$
- A subdivision map $\sigma_{\mathcal{R}}\colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$. $\sigma_{\mathcal{R}}$ is cellular and takes each open cell homeomorphically onto an open cell.

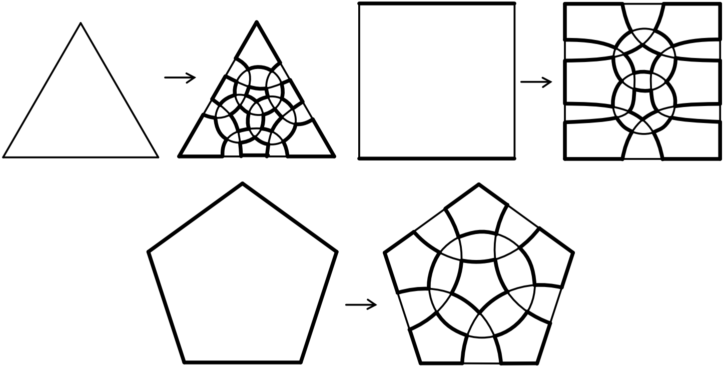
Definition of a finite subdivision rule $\mathcal{R}$

- C-F-P, Finite subdivision rules, *Conform. Geom. Dyn.* **5** (2001) 153–196
- finite CW complex $S_{\mathcal{R}}$ (called the *model subdivision complex*)
- $S_{\mathcal{R}}$ is the union of its closed 2-cells. Each 2-cell is modeled on a polygon (called a *tile type*). The 1-cells in $S_{\mathcal{R}}$ are called *edge types*.
- subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$
- A subdivision map $\sigma_{\mathcal{R}} \colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$. $\sigma_{\mathcal{R}}$ is cellular and takes each open cell homeomorphically onto an open cell.

- $\mathcal{R}$-*complex*: a 2-complex X which is the closure of its 2-cells, together with a map $h \colon X \to S_{\mathcal{R}}$ (called a *structure map*) which takes each open cell homeomorphically onto an open cell.
- One can use a finite subdivision rule to recursively subdivide $\mathcal{R}$-complexes. $\mathcal{R}(X)$ is the subdivision of X.

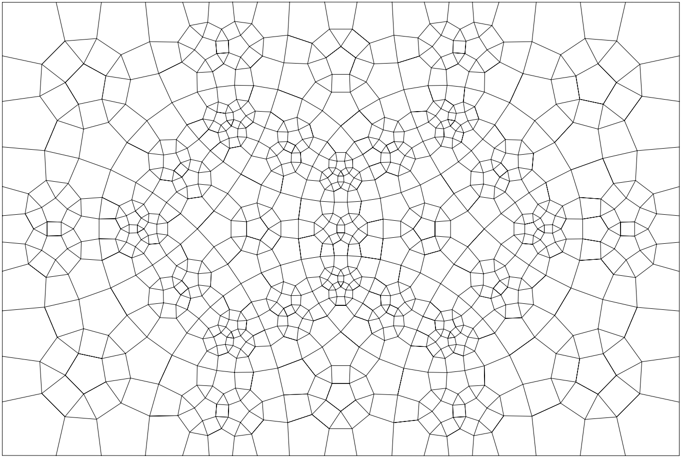
Example. The dodecahedral subdivision rule

The model subdivision complex has one vertex, two edges, and three tiles (a triangle, a quadrilateral, and a pentagon), and is hard to draw. Here are the subdivisions of the three tile types.

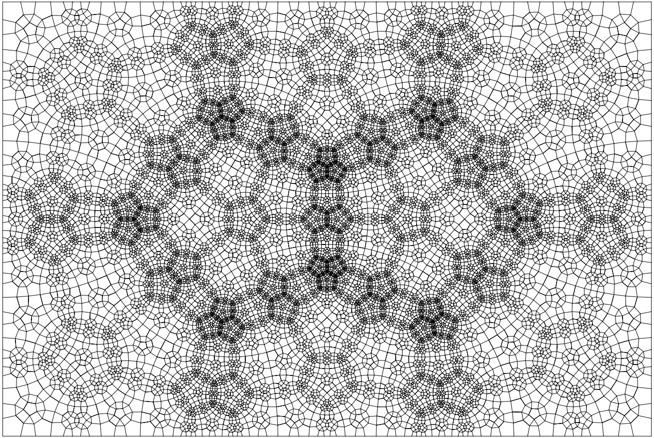


The second subdivision of the quadrilateral tile type

(The subdivision is drawn using Stephenson's CirclePack).

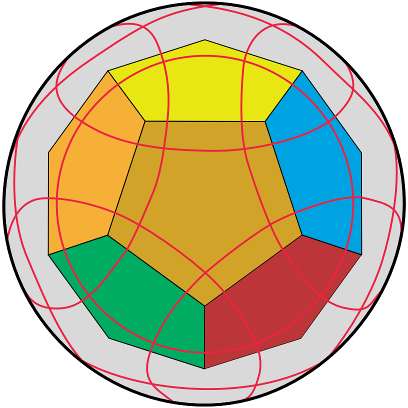


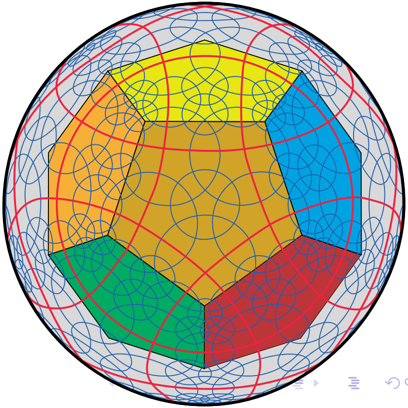
The third subdivision of the quadrilateral tile type



This subdivision rule on the sphere at infinity

The dodecahedral subdivision rule comes from the recursion at infinity for a Kleinian group generated by the reflections in a right-angled dodecahedron (image from SnapPea). Each face is in a hyperbolic plane which meets the boundary sphere in a red circle. The image at the right shows the circles at infinity for faces of the star of the fundamental region.





Motivation from the 1970's

Mostow's Rigidity Theorem (special case): If two closed hyperbolic n-manifolds, $n \geq 3$, have isomorphic fundamental groups, then they are isometric.

Thurston's Hyperbolization Conjecture: If M is a closed 3-manifold such that $\pi_1(M)$ is infinite, is not a free group, and does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, then M has a hyperbolic structure.

Motivation from the 1970's

Mostow's Rigidity Theorem (special case): If two closed hyperbolic n-manifolds, $n \geq 3$, have isomorphic fundamental groups, then they are isometric.

Thurston's Hyperbolization Conjecture: If M is a closed 3-manifold such that $\pi_1(M)$ is infinite, is not a free group, and does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, then M has a hyperbolic structure.

- A key ingredient of the proof of Mostow's theorem is the action of the fundamental group on the "boundary" of hyperbolic space.
- If you were given a group (say from a presentation) that was the fundamental group of a closed hyperbolic 3-manifold, could you recover the hyperbolic manifold from the group?
- Can you define the boundary of a group?
- Can you tell when the boundary of a group is a topological 2-sphere?

Cannon's Conjecture

Cannon's Conjecture: If G is a Gromov-hyperbolic discrete group whose space at infinity is S^2, then G acts properly discontinuously, cocompactly, and isometrically on $\mathbb{H}^3$.

- While a primary motivation for this was Thurston's Hyperbolization Conjecture, even after Perelman's proof of the Geometrization Conjecture this conjecture is still open.
- How do you proceed from combinatorial/topological hypotheses to an analytic conclusion?

Cannon's Conjecture

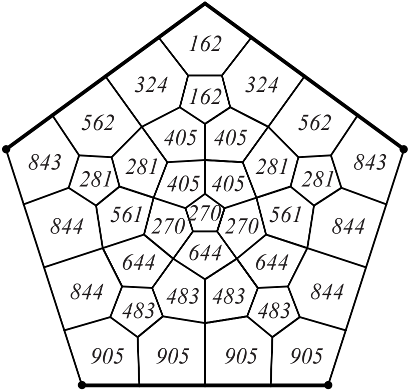
Cannon's Conjecture: If G is a Gromov-hyperbolic discrete group whose space at infinity is S^2, then G acts properly discontinuously, cocompactly, and isometrically on $\mathbb{H}^3$.

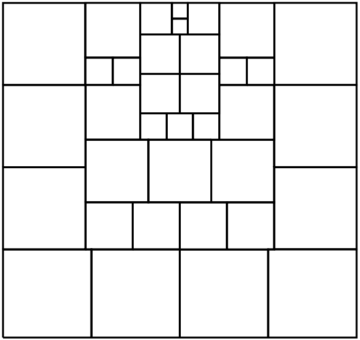
- While a primary motivation for this was Thurston's Hyperbolization Conjecture, even after Perelman's proof of the Geometrization Conjecture this conjecture is still open.
- How do you proceed from combinatorial/topological hypotheses to an analytic conclusion?
- Given a sequence of subdivisions of a tiling (or a shingling), how do you understand/control the shapes of tiles?
- When can you realize the subdivisions so that the subtiles stay almost round? (You don't need almost roundness, but it guarantees that the following two axioms are satisfied.)

Weight functions, combinatorial moduli

- *shingling* (locally-finite covering by compact, connected sets) $\mathcal{T}$ on a surface S, ring (or quadrilateral) $R \subset S$
- *weight function* ρ on $\mathcal{T}$: $\rho\colon\ \mathcal{T} \to \mathbb{R}_{\geq 0}$
- ρ*-length* of a curve, ρ*-height* H_ρ of R, ρ-area A_ρ of R, ρ*-circumference* C_ρ of R
- *moduli* $M_\rho = H_\rho^2/A_\rho$ and $m_\rho = A_\rho/C_\rho^2$
- *fat flow modulus* $M(R) = \sup_\rho H_\rho^2/A_\rho$ and *fat cut modulus* $m(R) = \inf_\rho A_\rho/C_\rho^2$
- The sup and inf exist, and are unique up to scaling. (This follows from compactness and convexity.)

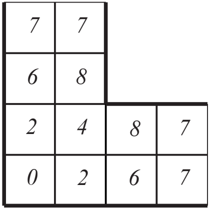
Optimal weight functions - an example

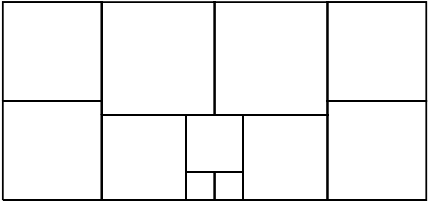


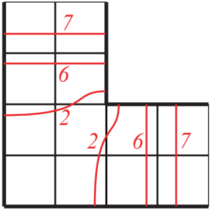


Optimal weight functions - another example

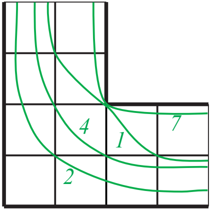
The optimal weight function is a linear combination of weight functions from fat flows and of weight functions from skinny cuts. This is why you get a squaring.







fat flows



skinny cuts

Combinatorial Riemann Mapping Theorem

- Now consider a sequence of shinglings of S.
- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli of rings
- **Axiom 2.** Existence of local rings with large moduli
- *conformal sequence* of shinglings: Axioms 1 and 2, plus mesh locally approaching 0.

Combinatorial Riemann Mapping Theorem

- Now consider a sequence of shinglings of S.
- **Axiom 1.** Nondegeneration, comparability of asymptotic combinatorial moduli of rings
- **Axiom 2.** Existence of local rings with large moduli
- *conformal sequence* of shinglings: Axioms 1 and 2, plus mesh locally approaching 0.

Theorem (C): If $\{\mathcal{S}_i\}$ is a conformal sequence of shinglings on a topological surface S and R is a ring in S, then R has a metric which makes it a right-circular annulus such that analytic moduli and asymptotic combinatorial moduli on rings in R are uniformly comparable.

- So there is a quasiconformal structure on S with analytic moduli uniformly comparable to asymptotic combinatorial moduli.
- J. W. Cannon, The combinatorial Riemann mapping theorem, *Acta Math.* **173** (1994), 155–234.

The Cannon-Swenson Theorem

Theorem (C-Swenson): In the setting of Cannon's conjecture, it suffices to prove that the sequence $\{\mathcal{D}(n)\}_{n\in\mathbb{N}}$ of disks at infinity is conformal. Furthermore, the $\mathcal{D}(n)$'s satisfy a linear recursion.

- J. W. Cannon, E. L. Swenson, Recognizing constant curvature groups in dimension 3, *Trans. Amer. Math. Soc.* **350** (1998), 809–849.

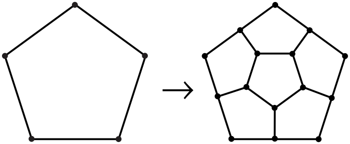
The Cannon-Swenson Theorem

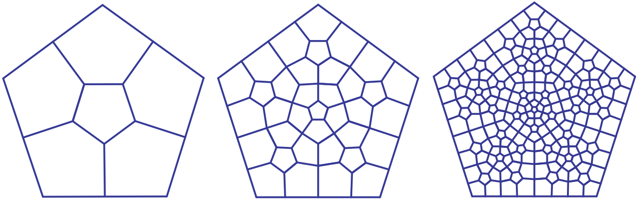
Theorem (C-Swenson): In the setting of Cannon's conjecture, it suffices to prove that the sequence $\{\mathcal{D}(n)\}_{n\in\mathbb{N}}$ of disks at infinity is conformal. Furthermore, the $\mathcal{D}(n)$'s satisfy a linear recursion.

- J. W. Cannon, E. L. Swenson, Recognizing constant curvature groups in dimension 3, *Trans. Amer. Math. Soc.* **350** (1998), 809–849.
- Finite subdivision rules were created as toy models for the sequences of covers by disks at infinity.

The pentagonal expansion complex

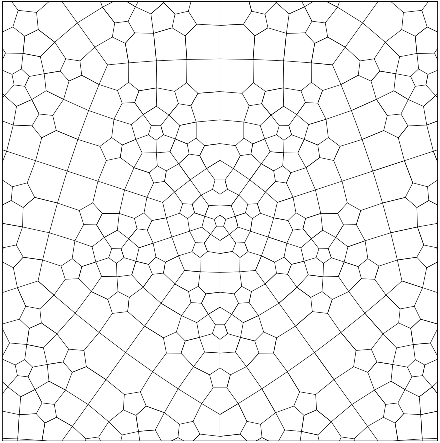
- Bowers and Stephenson defined the pentagonal expansion complex as the direct limit of the sequence of subdivisions of the tile type of the pentagonal subdivision rule.
- P.L. Bowers and K. Stephenson, A “regular” pentagonal tiling of the plane, *Conform. Geom. Dyn.* **1** (1997) 58–68





The pentagonal expansion complex

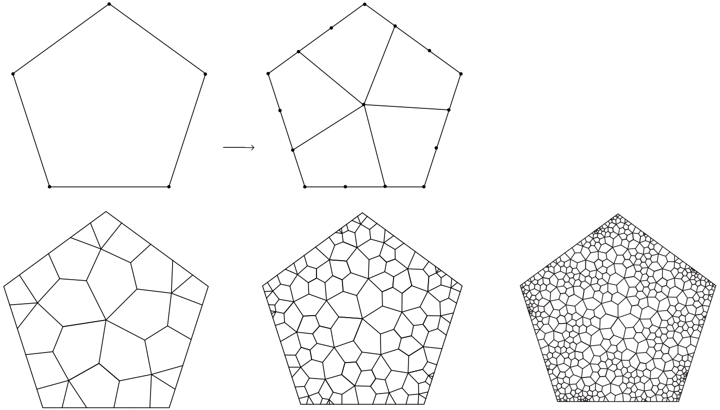
- They put a conformal structure on the expansion complex so that each tile is conformally a regular pentagon. They showed that the expansion map is conformal, and hence that the expansion complex is parabolic.



Application of expansion complexes

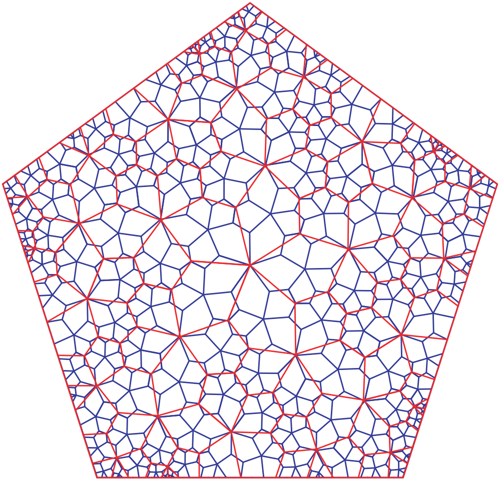
- More generally, an expansion $\mathcal{R}$-complex is an $\mathcal{R}$-complex X with structure map $f\colon X \to S_{\mathcal{R}}$ such that X is homeomorphic to $\mathbb{R}^2$ and there is an orientation-preserving homeomorphism $\varphi\colon X \to X$ (the expansion map) with $\sigma_{\mathcal{R}} \circ f = f \circ \varphi$.
- The dihedral symmetry for the pentagonal subdivision rule makes it much easier to show that the expansion map is conformal. You would like to be able to make use of rotational symmetry. The intuition is that for Cannon's Conjecture expansion complexes correspond to tangent spaces at infinity, and at fixed points of loxodromic elements you will see rotational (but not dihedral) symmetry.

An example with rotational symmetry



Superimposed subdivisions

Here are the third and fourth subdivisions, superimposed. Note the vertices.



The expansion complex

- One can put a piecewise conformal structure on the expansion complex X with regular pentagons, and then use power maps to extend over the vertices. (This is inspired by the Bowers-Stephenson construction.)
- The expansion map agrees with a conformal map on the vertices. One can conjugate to get a new fsr for which this conformal map is the expansion map. The subdivision map is conformal with respect to the induced conformal structure on the subdivsion complex.
- C-F-P, Expansion complexes for finite subdivision rules I, *Conform. Geom. Dyn.* **10** (2006) 63–99
  C-F-P, Expansion complexes for finite subdivision rules II, *Conform. Geom. Dyn.* **10** (2006) 326–354

Conformal tilings

- More recently, Bowers and Stephenson have been building a more general theory of expansion complexes where you do not require that there is a single expansion map.
- In our setting, this would correspond to tangent spaces at infinity for points that are not fixed points of loxodromic elements.
- Philip L. Bowers and Kenneth Stephenson. Conformal tilings I: foundations, theory, and practice. *Conform. Geom. Dyn.* **21** (2017) 1–63
  Philip L. Bowers and Kenneth Stephenson. Conformal tilings II: Local isomorphism, hierarchy, and conformal type. *Conform. Geom. Dyn.* **23** (2019) 52–104

Two questions

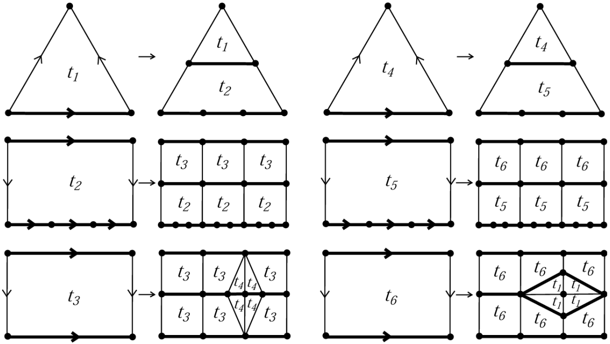
- In the setting of Cannon's conjecture, can you have hyperbolic expansion complexes? (Unfortunately the type problem is hard. Can a finite subdivision rule have hyperbolic and parabolic expansion complexes that are locally isomorphic?)

Two questions

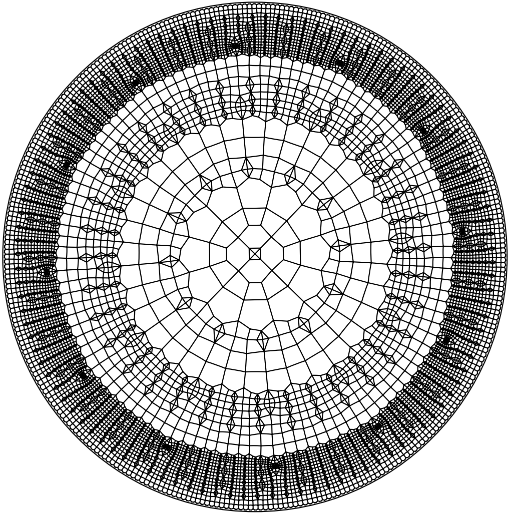
- In the setting of Cannon's conjecture, can you have hyperbolic expansion complexes? (Unfortunately the type problem is hard. Can a finite subdivision rule have hyperbolic and parabolic expansion complexes that are locally isomorphic?)
- In a parabolic expansion complex, how nice is the asymptotic shape of a tile (or of the seed of an expansion complex)?

A finite subdivision rule with hyperbolic and parabolic expansion complexes

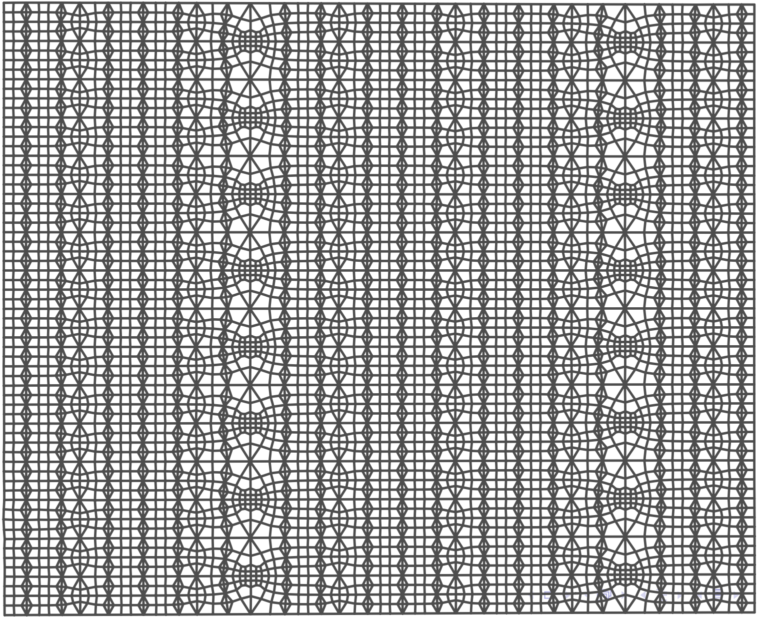
- The subdivisions of the six tile types.
- The following expansion complexes are locally isomorphic, so any compact subcomplex of one is isomorphic to subcomplexes of the others.



Part of a hyperbolic expansion complex



Part of a parabolic expansion complex Y



Is this expansion complex hyperbolic or parabolic?